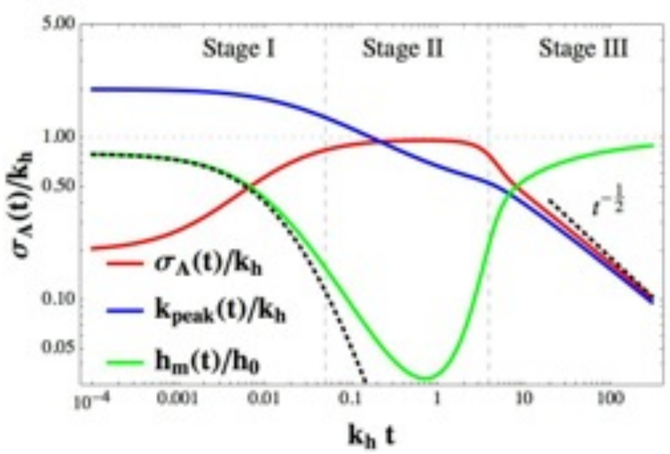
The self-similar evolution of inverse cascade of magnetic helicity driven by anomaly



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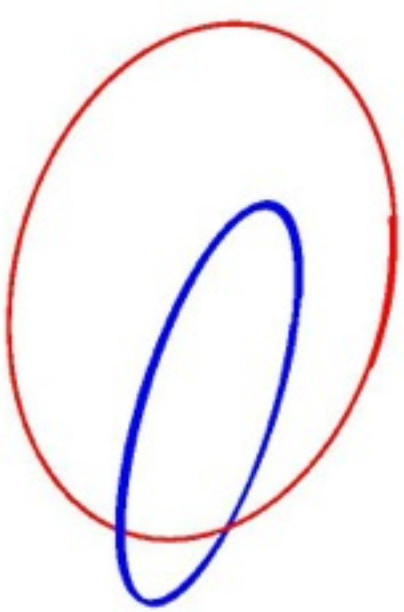
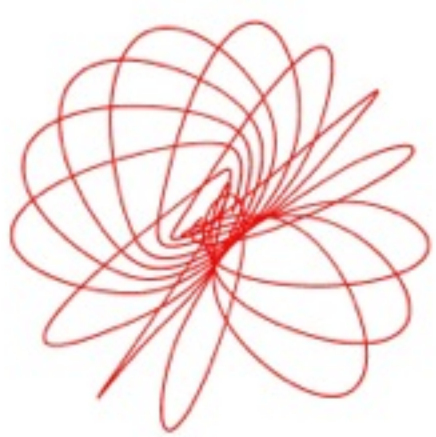




Based on: Y. Hirono, D. Kharzeev and YY, in preparation.

RBRC lunch seminar,

BNL, Jun. 18th

Chiral magnetic effect and chiral anomaly

- The generation of electric-magnetic (EM) current in a plasma with chiral fermions along the direction of magnetic field:

$$\mathbf{j}_{\rm CME} = C_A \mu_A \mathbf{B} = \sigma_A \mathbf{B}\,, \qquad \partial_\mu j^\mu_A = C_A \mathbf{B}\cdot\mathbf{E}\,,$$

  where anomaly coefficient is $C_A = (N_c e^2)/(2\pi^2)$ and μ_A the axial charge potential. σ_A is refered as chiral magnetic conductivity.

- Nielsen-Ninomia argument (1983):

$$\mu_A \frac{dn_A}{dt} = \mu_A C_A \mathbf{B}\cdot\mathbf{E} = \mathbf{j}_{\rm CME}\cdot\mathbf{E}\,.$$

  Energy needed to remove one chiral fermion is balanced by work done by the CME current.

- CME might play important roles in various physical situations, including heavy-ion collisions (Kharzeev-McLerran-Warringa, 2007), astrophysical physics (Vilenkin, 1980) and condensed matter system (Nielsen-Ninomia, 1983).

Dynamical evolution of a chiral plasma

- The evolution of axial charge density and EM fields are coupled due to CME and anomaly relation.

- Due to Ampere's law, a CME current will induce EM field.

$$\nabla \times \mathbf{B} - \frac{\partial \mathbf{E}}{\partial t} = \mathbf{j}_{\mathrm{EM}} = \sigma_A \mathbf{B} + \sigma \mathbf{E}\,,$$

- EM fields will back-react on the axial charge via anomaly equation:

$$\partial_t n_A = C_A \mathbf{B} \cdot \mathbf{E}\,.$$

- What would be the fate of EM fields and axial charge density (or σ_A) under such coupled evolution? Any universal feature of such evolutions?

Conservation of total helicity and inverse cascade of magnetic helicity

- We consider magnetic heilcity h_m and define "fermonic helicity" h_F:

$$h_m \equiv \int d^3x \mathbf{A} \cdot \mathbf{B} \,, \qquad h_F \equiv C_{\mathrm{A}}^{-1} \int d^3x \, n_A \,.$$

- Total helicity of the system is conserved: $h_0 \equiv h_m + h_F = \text{const}$:

$$\partial_t h_m(t) = -\int d^3x \mathbf{E} \cdot \mathbf{B} = -C_A^{-1} \partial_t \int d^3x n_A = -\partial_t h_F(t) \,.$$

- Guiding principle: the system would prefer to preserve helicity while minimize energy cost.
- Inverse cascade of magnetic helicity: A " small scale magnetic field" would evolve into a " large scale magnetic field" which has a smaller energy per unit helicity (qualitative argument in the context of astro-physics, Joyce-Shaposhnikov, 1997).
- This work: dynamical realization of such inverse cascade and role played by anomaly.

Helicity spectrum I

- To introduce magnetic helicity spectrum, we consider eigen-functions (states) of curl operator:

$$\nabla \times \mathbf{V}_{\pm}(\mathbf{x}; k) = \pm k \mathbf{V}_{\pm}(\mathbf{x}; k)\,.$$

  Those states are refered as Chardrasekhar-Kendall (CK) states in literature (Chardrasekhar-Kendall, 1957).

- In an open space, CK states are just ploarized plane waves. We consider CK states $W^{\pm}_{lm}(\mathbf{x}; k)$ in a spherical symmetric domain:

$$\nabla \times \mathbf{W}^{\pm}_{lm}(\mathbf{x}; k) = \pm k \mathbf{W}^{\pm}_{lm}(\mathbf{x}; k)\,.$$

  where $l = 0, 1, \dots, m = -l, -l+1, \dots l$. $W^{\pm}_{lm}(\mathbf{x}; k)$ can be expressed in terms of spherical harmonics and spherical Bessel functions (and can be found in Jackson's textbook).

Helicity spectrum II

- We expand magnetic field $\mathbf{B}$ in terms of CK states $\mathbf{W}^{\pm}_{lm}(\mathbf{x}, ; k)$:

$$\mathbf{B}(\mathbf{x}, t) = \sum_{l,m} \int_0^\infty \frac{dk}{\pi} k^2 \left[\alpha^+_{lm}(k, t)\mathbf{W}^+_{lm}(\mathbf{x}; k) + \alpha^-_{lm}(k, t)\mathbf{W}^-_{lm}(\mathbf{x}; k)\right] ,$$

  and introduce magnetic helicity spectrum which measures the relative weight from one single mode $W^{\pm}_{lm}(\mathbf{x}; k)$:

$$g_{\pm}(k, t) \equiv \sum_{l,m} |\alpha^{\pm}_{lm}(k, t)|^2 .$$

- The magnetic helicity and energy of magnetic field can be expressed in terms of $g_{\pm}(k, t)$ as

$$\begin{aligned} h_m(t) &\equiv \int d^3x\, \mathbf{A}(\mathbf{x}, t) \cdot \mathbf{B}(\mathbf{x}, t) = \int_0^\infty \frac{dk}{\pi} k \left[g_+(k, t) - g_-(k, t)\right] , \\ \mathcal{E}_M(t) &\equiv \int d^3x\, \frac{1}{2}\mathbf{B}^2(\mathbf{x}, t) = \frac{1}{2}\int_0^\infty \frac{dk}{\pi} k^2 \left[g_+(k, t) + g_-(k, t)\right] . \end{aligned}$$

- A single CK state $W^+(\mathbf{x}, k)$ ($W^-(\mathbf{x}, k)$) carries positive (negative) helicity. The energy cost per unit helicity for a single CK state is k.

Inverse cascade driven by anomaly

- The energy cost per unit helicity for a CK state $W^{\pm}(\mathbf{x}, k)$ is given by k. The energy cost per unit helicity of a chiral fermion is $\sigma_A = C_A^{-1}\mu_A$.
- CK states with $k > \sigma_A$ will transfer magnetic helicity to fermonic helicity. Fermionic helicity will be transfered to soft magnetic modes $k < \sigma_A$.
- The peak of magnetic helicity spectrum, k_{peak}, will decrease, indicating inverse cascade.
- The system will eventually approach the CK state with the lowest possible $k = k_{\mathrm{min}}$ (equilibrium state).
- NB: it is convenient to introduce a characteristic energy scale k_h, the energy cost per unit helicity of the system if total helcity h_0 is completely carried by chiral fermion, i.e. , $h_0 = h_F$. The inverse cascade will not happen if $k_{\mathrm{min}} > k_h$.

Maxwell's theory in the presence of anomaly

- Maxwell's equation in the presence of chiral magnetic current gives:

$$\sigma\partial_t \mathbf{B}(t,\mathbf{x}) = \nabla^2\mathbf{B}(t,\mathbf{x}) + \sigma_A(t)\,(\nabla\times\mathbf{B})\ .$$

  We have neglected the spatial dependence of $n_A(t) \approx V^{-1}\int d^3x\, n_A(\mathbf{x},t)$ where V is the volume.

- In terms of $\alpha_{lm}(k,t)$:

$$\partial_t\alpha^{\pm}_{lm}(k,t) = \sigma^{-1}\left[-k^2 \pm \sigma_A(t)k\right]\alpha^{\pm}_{lm}(k,t)\,.$$

  The system is non-linear. $g_{\pm}(k,t)$ thus $h_m(t)$ can be computed from $\alpha^{\pm}_{lm}(k,t)$. $\sigma_A(t)$ will be determined from self-consistency condition $h_m(t) + h_F(t) = \text{const}$.

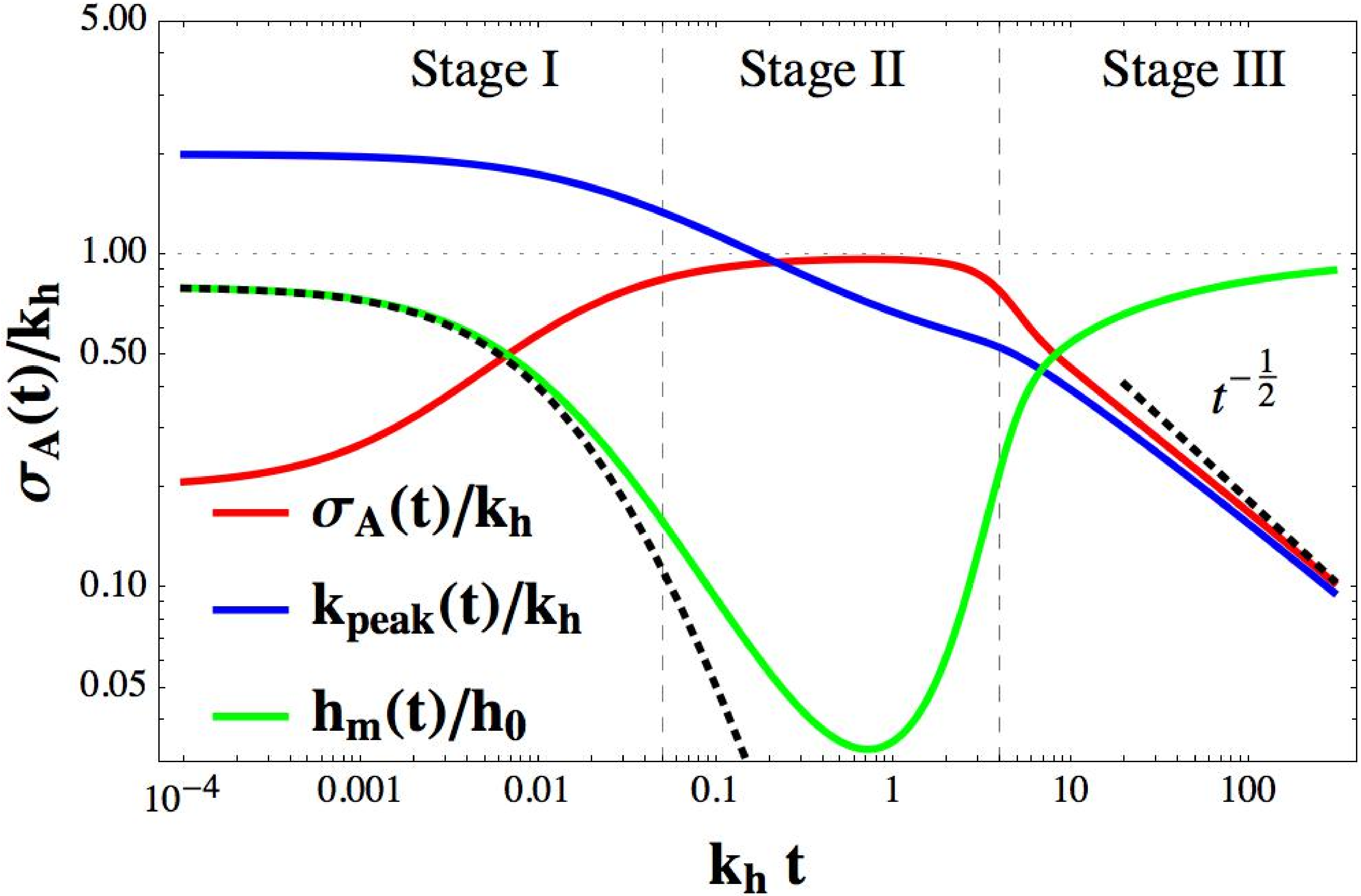
- Mode with $k > \sigma_A$ will decay exponentially. Instability: if $k < \sigma_A(t)$ (assuming $\sigma_A > 0$), $\alpha^{+}_{lm}(k,t)$ will grow exponentially (Joyce-Shaposhnikov, PRL, 1997).

- We will use Maxwell's theory in the presence of CME current + anomaly relation to study the inverse cascade of magnetic helicity.

Initial conditions

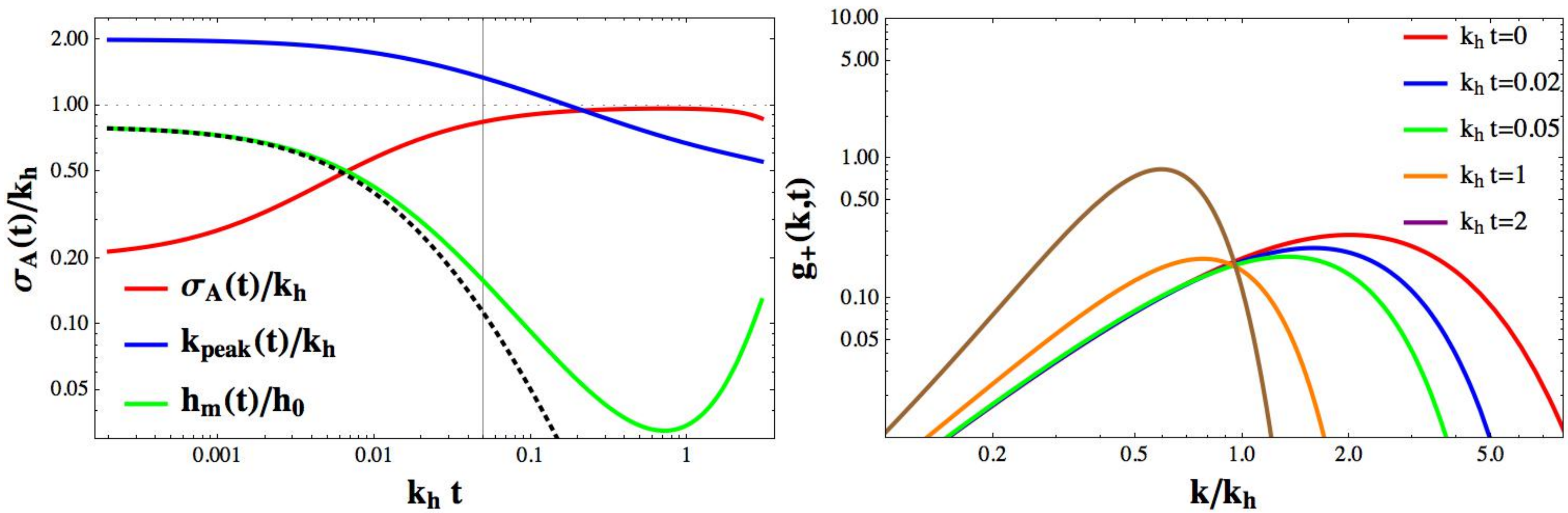
- The evolution of $\sigma_A(t)$, $\mathbf{B}(\mathbf{x}, t)$ and magnetic helicity function $g(t,k) = g_+(t,k) - g_-(t,k)$ depends on initial profile: $\mathbf{B}(\mathbf{x}; t = 0)$ and $\sigma_A(t = 0)$ (or equivalently $h_{F,I}/h_0$) .
- To illustrate inverse cascade, we consider the scenario that initially the peak of magnetic helicity spectrum $g_I(k)$, $k_{\rm peak} \gg k_{\rm min}$ and magnetic helicity is dominant over fermionic helcity.
- We take Hopfion solutions to vaccum Maxwell equation as the initial condition for EM field. Such solution carries non-zero helicity (which we assume to be positive) and finite energy. (We will visualize a Hopfion solution in the real space later)
- We will focus on the time dependence of $k_{\rm peak}(t)$, $\sigma_A(t)$, $h_m(t)/h_0$ and magnetic helicity spectrum $g(k,t)$.

Stages of evolution



- The evolution can be schematically divided into three stages.

Stage I and Stage II



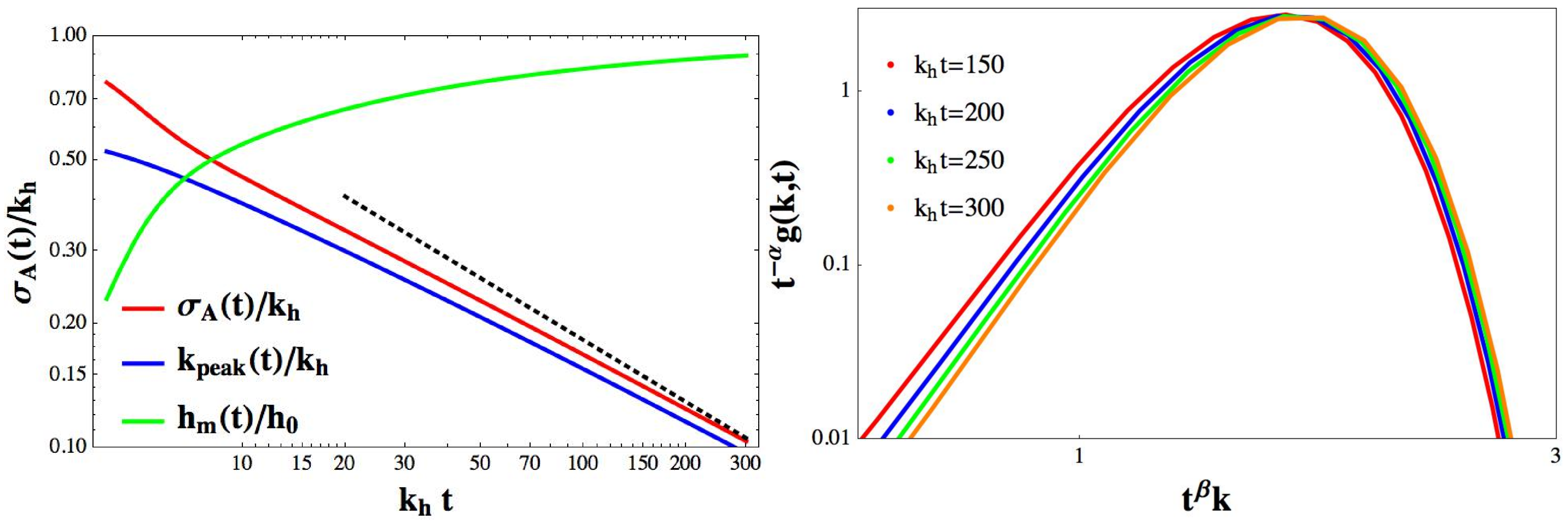
- Magnetic helicity $h_m(t)$ decays exponentially and is transferred to fermionic helicity. Meanwhile, $k_{\rm peak}(t)$ starts decreasing. The duration of "stage I", τ_I, is approximately:

$$\tau_I \sim \sigma k_{\rm peak}^{-2}(t=0)\,.$$

- Stage II: in this stage, total helicity $h_0 \approx h_F$. $k_{\rm peak}(t)$ continues decreasing. "Stage II" ends when $\kappa_{\rm peak}$ is close σ_A. The duration of this stage is:

$$\tau_{II} \sim \sigma k_h^{-2}\,.$$

Stage III: self-similar stage



- in this stage, both $\sigma_A(t)$ and $\kappa_{\rm peak}(t)$ decrease and $\sigma_A(t) \approx \kappa_{\rm peak}(t)$ $(\nabla \times \mathbf{B} \approx k_{\rm peak}(t)\mathbf{B} \approx \sigma_A(t)\mathbf{B})$. $h_m(t)$ will approach h_0. $\sigma_A(t), \kappa_{\rm p}(t)$ behave as a power law in t :

$$k_h(t) \approx \sigma_A(t) \propto t^{-\beta}\,.$$

Meanwhile, the evolution of $g(k,t)$ becomes self-similar:

$$g(k,t) \sim t^{\alpha}\tilde{g}(t^{\beta}k)\,, \qquad \alpha = 1\,, \qquad \beta = 1/2\,.$$

Scaling exponents

- As $h_m \approx$ const at late time, we have from that $\alpha = 2\beta$:

$$h_m(t) = \int \frac{dk}{\pi} k g(k) = \int \frac{dk}{\pi} k\, t^{\alpha} \tilde{g}(t^{\beta} k) = t^{\alpha - 2\beta} \int \frac{dx}{\pi} \tilde{g}(x) \approx \text{const}\,.$$

- From Maxwell's equations, one would have

$$g^{\pm}(k,t) = g_I^{\pm}(k) \exp\left\{ 2\sigma^{-1} \left[-k^2 t \pm k \int_0^t dt'\, \sigma_A(t') \right] \right\}.$$

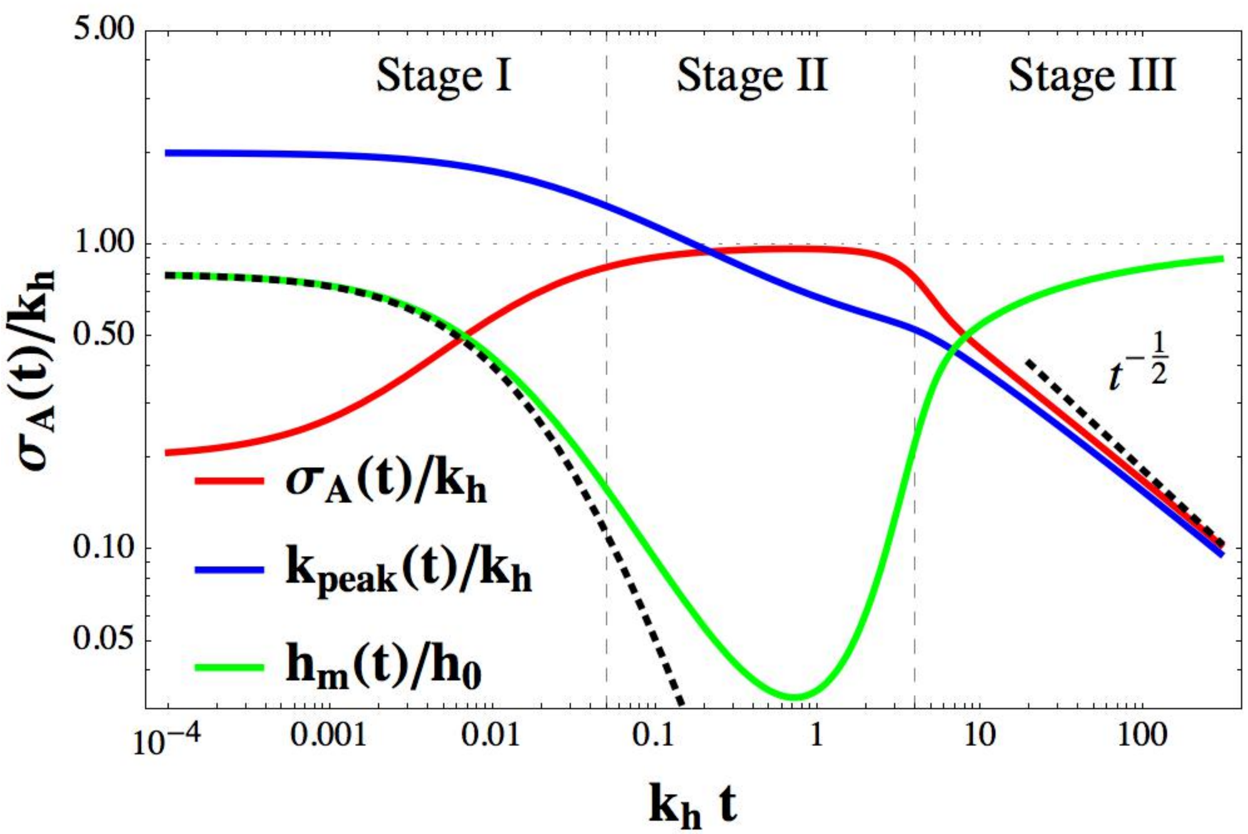
  Matching to scaling form $\tilde{g}(t^{\beta} k)$ gives:

$$\beta = \frac{1}{2}\,, \qquad \sigma_A(t) \sim t^{\beta - 1}\,.$$

- Using $k_{\rm peak}(t) \sim \sigma_A(t)$, we have:

$$g(k,t) \propto \exp\{-2\sigma^{-1} \left[k - k_{\rm p}(t) \right]^2 t\} \to \delta(k - k_{\rm peak}(t))\,.$$

  During self-similar evolution, the width of Gaussian becomes narrower and narrower and magnetic field is close to a single CK state $W^{+}(k_{\rm peak}(t), t)$.



- We start with a "small scale magnetic field" and follow its evolution.
- The magnetic helicity is first transfered to fermionic helicity and later ferminoic helicity is transfered into magnetic helicity.
- $k_{\rm peak}(t)$ decreases, indicating inverse cascade.
- The system spends a long time in self-similar stages.

Hopfions solutions

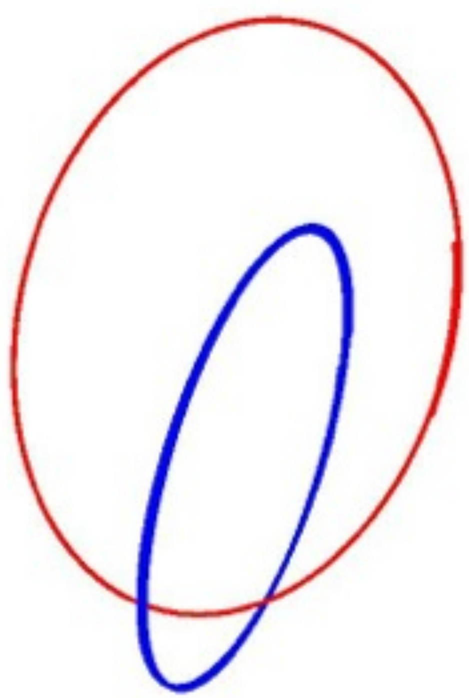
- Hopfion solutions might be interpreted as a soliton wave solution to Maxwell's equation.

$$\mathbf{B}_{\mathrm{Hopf}}(\mathbf{x},t) \propto \sqrt{\frac{4}{3\pi}}\int_0^\infty dk k^2 e^{-kL_{\mathrm{Hopf}}}\left[(kL^2_{\mathrm{Hopf}})\mathbf{W}^+_{11}(\mathbf{x};k)e^{-ikt}+\text{c.c.}\right],$$
$$\mathbf{E}_{\mathrm{Hopf}}(\mathbf{x},t) \propto \sqrt{\frac{4}{3\pi}}\int_0^\infty dk k^2 e^{-kL_{\mathrm{Hopf}}}\left[(-ikL^2_{\mathrm{Hopf}})\mathbf{W}^+_{11}(\mathbf{x};k)e^{-ikt}+\text{c.c.}\right]$$

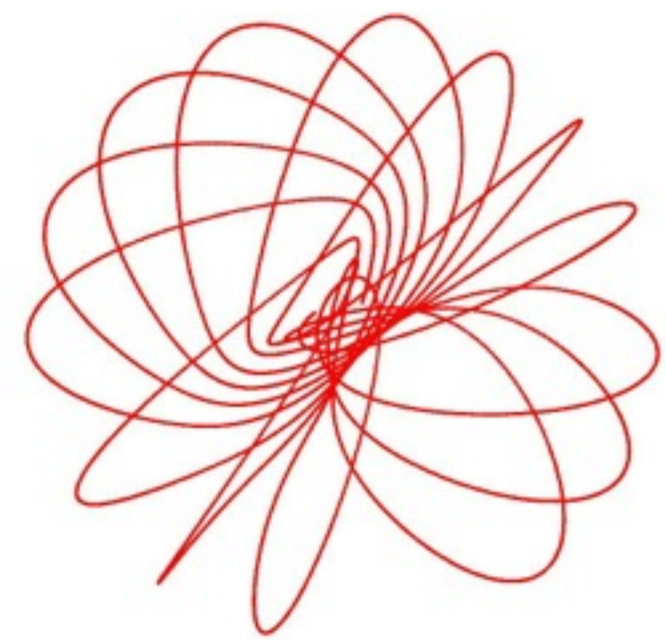
where L_{Hopf} controls the size of a Hopfion solution.

Hopfion vs CK state

- Magnetic helicity counts number of linking and knots.
- We have studied the evolution of magnetic field from a Hopfion configuration towards a single CK states.
- For a Hopfions solutions, field lines form closed loops and are linked. A single CK state is “knotted”.

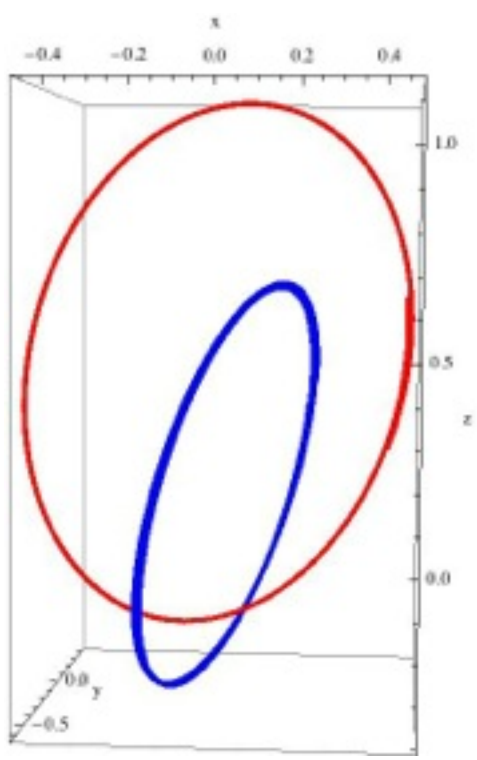


Hopfion

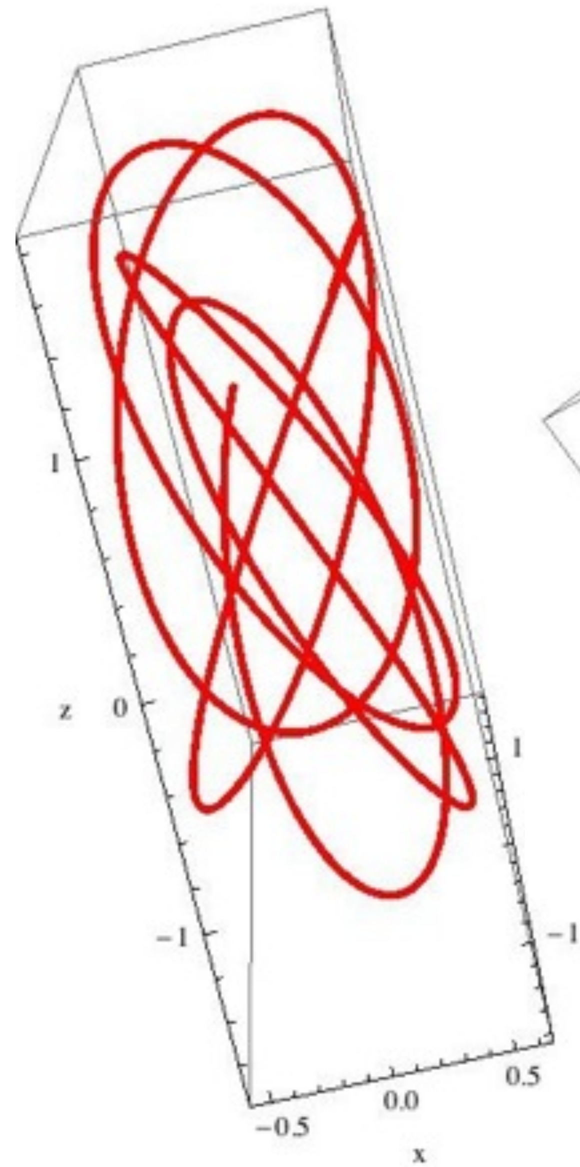


A single CK state

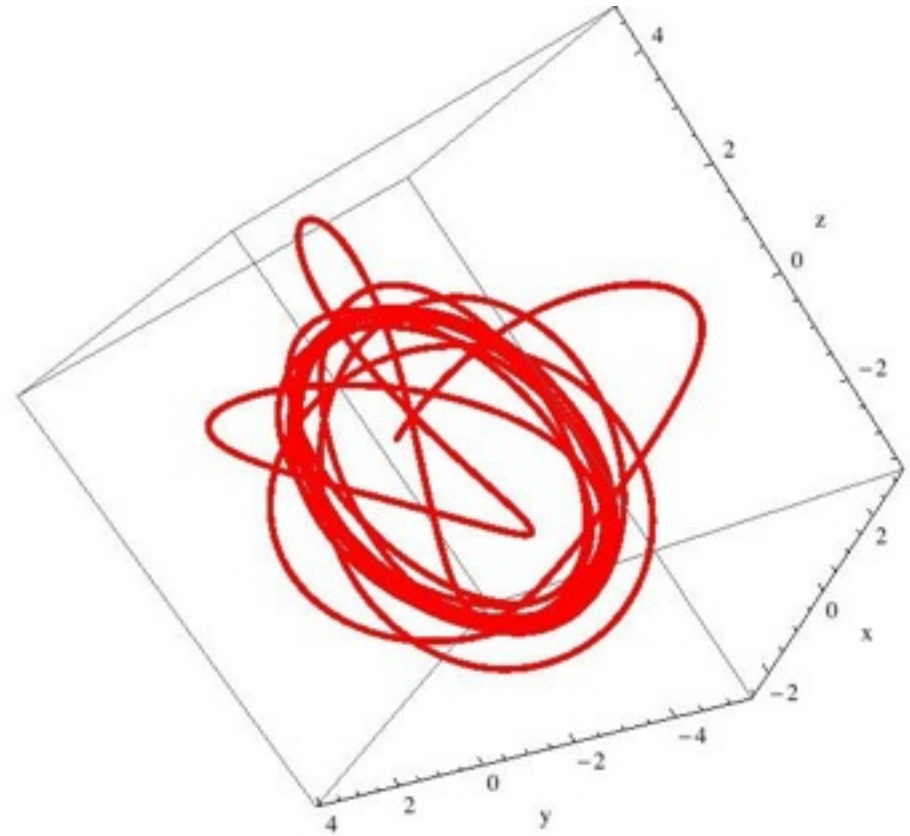
Snapshot of field lines during the evolution



Initial time



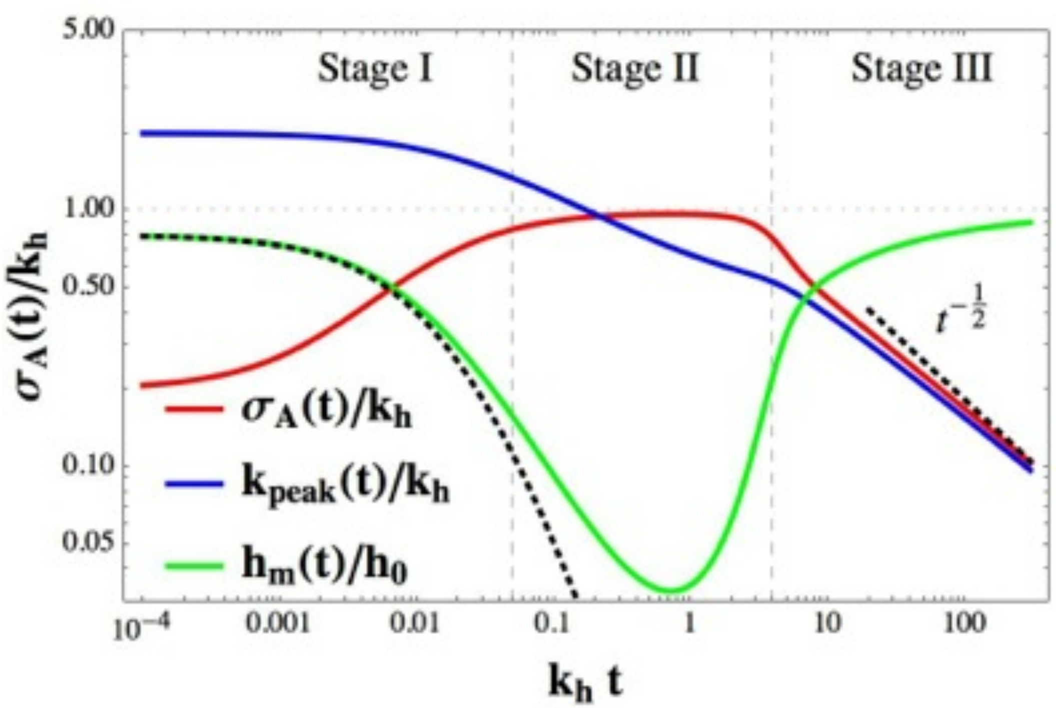
Stage I

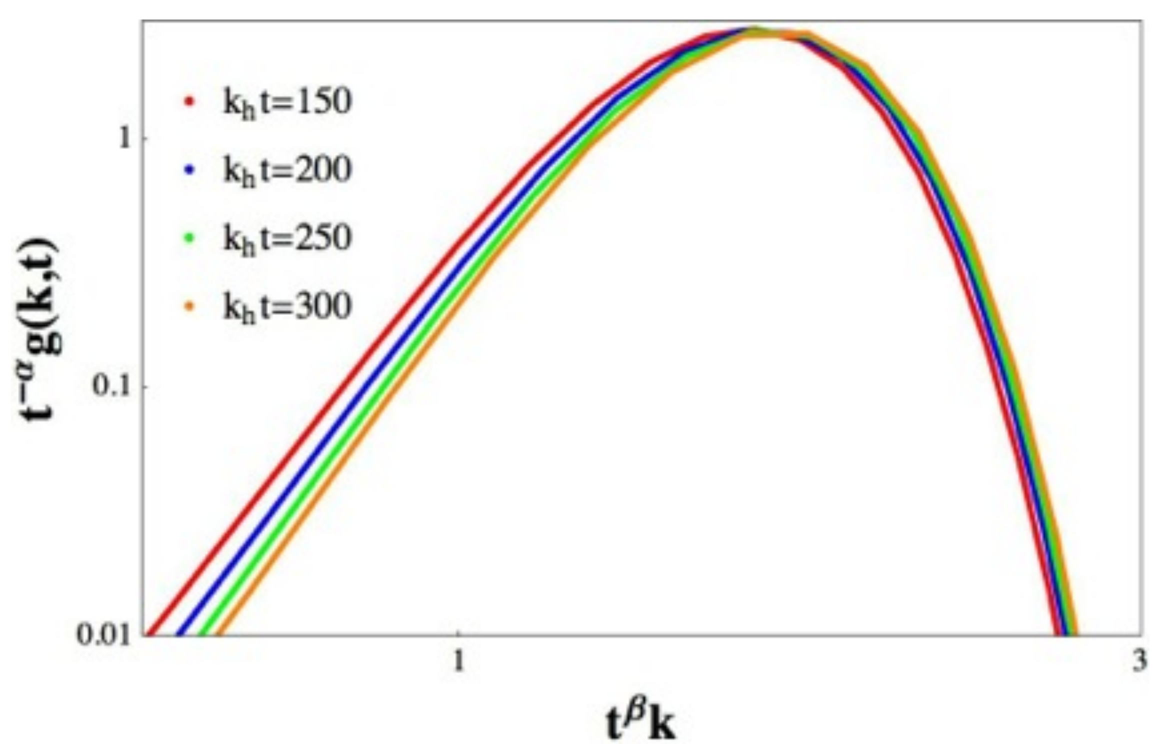


Stage II

Summary

- We have studied the inverse cascade of magnetic helicity driven by anomaly.
- Due to interplay between EM field and chiral anomaly, the evolution at late stage is self-similar.

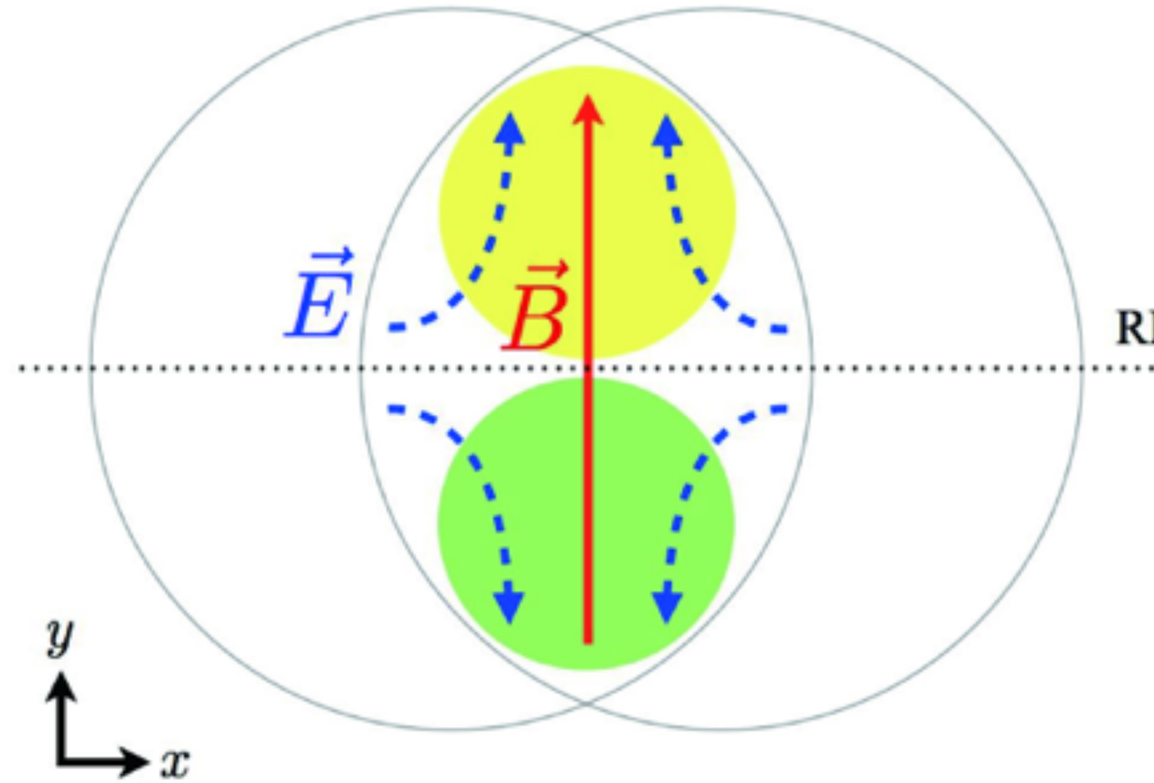




Any signal in heavy-ion collisions experiment?

- Helical EM fields are created by spectators. According to the scenario of inverse cascade, they would evolve into "a large scale" magnetic field and emit soft polarized photons at freeze-out time.

- The polarization of photons depends on the azimuthal angle.



- To satisfy $k_h > k_{\min} \sim R^{-1}$, we need to have large initial magnetic helicity, which corresponds to very strong local EM fields $eB \sim 30 m_\pi^2$, which might be realized by event-by-event fluctuations.

Outlook

- Extension to anomalous magneto-hydrodynamics (interplay among magnetic helicity, fermionic helicity and kinetic helicity).

- Generalization to non-Abelian theories (personal conjecture): would inverse cascade affect the generation of topological charges? would the long time limit $\langle Q_W^2 \rangle \propto V t^n, \qquad n = 1$ be modified.